## Circular Motion

---

Key Ideas

- An object is in uniform circular motion if it follows a circular path with a constant speed. Its velocity is changing because its direction is changing as it follows the circular path.
- Centripetal acceleration $\vec{a}_c$ is the acceleration a particle must have to follow a circular path. Centripetal acceleration is always directed toward the center of rotation and has magnitude $a_c = v^2/r$.
- Nonuniform circular motion occurs when there is tangential acceleration $\vec{a}_T$ of an object executing circular motion and the speed of the object is changing.

Learning Objectives

After completing this section, you should be able to...

- determine the centripetal acceleration of an object moving on a circular path,
- use the equations of circular motion to find the position, velocity, and acceleration of a particle executing circular motion,
- explain the differences between the centripetal acceleration and the tangential acceleration resulting from nonuniform circular motion, and
- evaluate centripetal and tangential acceleration in nonuniform circular motion to find the total acceleration vector.

Uniform circular motion is a specific type of motion in which an object travels along a circle or a portion of a circle with a constant speed. For example, any point on a propeller spinning at a constant rate is in uniform circular motion. Other examples are the second, minute, and hour hands of a watch. It is remarkable that points on these rotating objects are actually accelerating, although the rotation rate is a constant. This is illustrated in the following figure, where a three-blade propeller is shown. Three points on the blades are an equal distance r from the axis of rotation and follow the circular path indicated. The velocities of those points, which are tangent to the circle, all have equal magnitude, but their directions are continually changing with the constant rotation of the propeller.
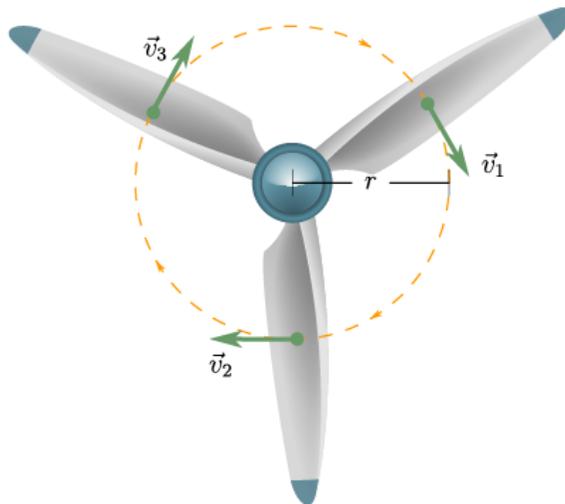
Figure 4.19 Points that are equidistant from the center of rotation for this propeller have velocities of equal magnitude

Centripetal Acceleration

In one-dimensional kinematics, objects with a constant speed have zero acceleration. However, in two- and three-dimensional kinematics, even if the speed is a constant, a particle can have acceleration if it moves along a curved trajectory such as a circle. In this case the velocity vector is changing, or $d\vec{v}/dt \neq 0$. This is shown in Figure 4.19. As the particle moves clockwise during an elapsed time Δt on the circular path, its position vector moves from $\vec{r}(t)$ to $\vec{r}(t + \Delta t)$. The velocity vector has constant magnitude and is tangent to the path as it changes from $\vec{v}(t)$ to $\vec{v}(t + \Delta t)$, changing its direction only.

In Cartesian coordinates, the position vector is given by $\vec{r}(t) = x\hat{i} + y\hat{j}$. The magnitude of the position vector is r, so we write

$$\vec{r}(t) = r \, \cos \, \theta \, \hat{i} + r \, \sin \, \theta \, \hat{j} \qquad \text{4.22}$$

To find the velocity, we take the derivative of the position vector with respect to time. Although r and the Cartesian unit vectors are constants and are independent of time, θ **does** depend on time, as do functions of θ.
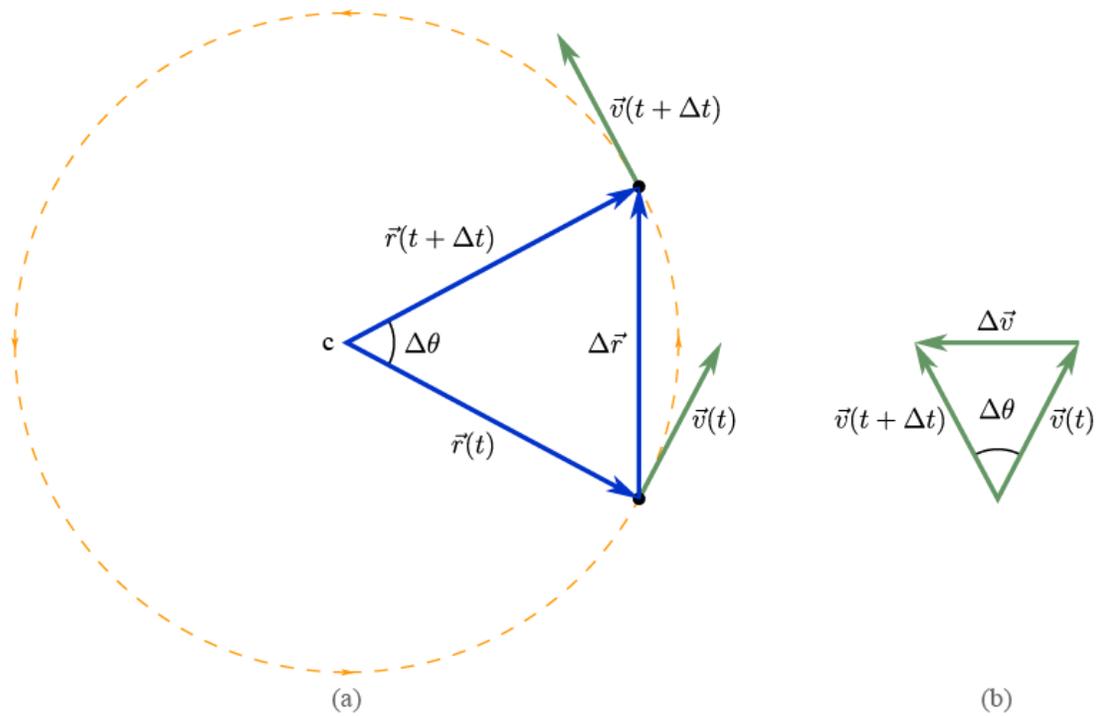
Recall that, if θ is in radians:

$$\frac{d}{d\theta}(\sin \, \theta) = \cos \, \theta \, \frac{d\theta}{dt} \qquad \text{and} \qquad \frac{d}{d\theta}(\cos \, \theta) = -\sin \, \theta \, \frac{d\theta}{dt}$$

We define $\omega = \frac{d\theta}{dt}$ as the **angular velocity**, which has units of radians per second (rad/s) and is the number of radians of angular measure through which the particle passes per second. If T is the **period** of motion, or the time to complete one revolution (2π rad/s), then $\omega = \frac{2\pi}{T}$.

Using this definition, we can express the **tangential velocity** of the particle as

$$\begin{aligned} \vec{v}_T &= \frac{d}{dt}\left(r \, \cos \, \theta \, \hat{i} + r \, \sin \, \theta \, \hat{j}\right) \\ &= -r\omega \, \sin \, \theta \, \hat{i} + r\omega \, \cos \, \theta \, \hat{j} \\ &= r\omega \left(-\sin \, \theta \, \hat{i} + \cos \, \theta \, \hat{j}\right) \end{aligned} \qquad \text{4.23}$$

Figure 4.20 (a) A particle undergoing uniform circular motion as it moves from position $\vec{r}(t)$ to position $\vec{r}(t + \Delta t)$ as it follows its circular path. The particle has an angular displacement $\Delta\theta$. (b) The particle's velocity changes direction during the elapsed time, but its magnitude is constant.

We see that the speed of the particle is given by $v = r\omega$. To find the acceleration, we take an additional derivative with respect to time of the velocity vector we just found in Equation 4.23. Since we are considering the case of constant speed around a circle here, both r and v are constants. Since $v = r\omega$, it follows that ω is a constant as well. Therefore,

$$
\begin{aligned}
\vec{a}(t) \quad &= \frac{d}{dt}\vec{v}(t) \\
&= r\omega\frac{d}{dt}\left(-\sin\,\theta\,\hat{i} + \cos\,\theta\,\hat{j}\right) \\
&= r\omega^2\left(-\cos\,\theta\,\hat{i} - \sin\,\theta\,\hat{j}\right)
\end{aligned}
$$

$$\boxed{4.24}$$

We see that the acceleration of the particle has a magnitude of $r\omega^2$. Since $v = r\omega$, we can also write the acceleration magnitude as v^2/r, which has the following special name.
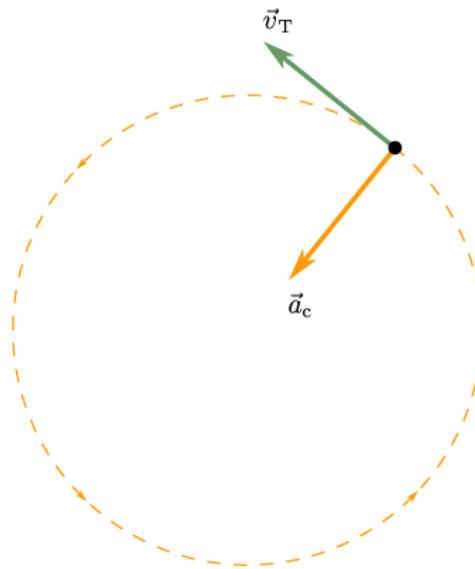
Centripetal Acceleration

The acceleration of a particle following a circular path of radius r at constant speed v is called the centripetal acceleration. It is directed toward the center of the circle and has the following magnitude.

$$a_c = \frac{v^2}{r} \qquad\qquad \boxed{4.25}$$

Alternatively,

$$\vec{a}_c = -r\omega^2\,\hat{r} \qquad\qquad \boxed{4.26}$$

Figure 4.21 A particle is moving in uniform circular motion. The particle's velocity is tangent to the circle and its centripetal acceleration is perpendicular to the tangential velocity and points toward the center of the circle.

The direction of the acceleration vector is perpendicular to the velocity, pointing toward the center of the circle (Figure 4.18), so we can write $\vec{a}(t) = -\omega^2 \vec{r}(t)$. This is a radial acceleration and is called the **centripetal acceleration**, which is why we give it the subscript c. The word centripetal comes from the Latin words *centrum* (meaning "center") and *petere* (meaning "to seek"), and thus takes the meaning "center seeking."

End of Content Preview.

Request access to a full digital copy of the at www.theexpertta.com/univphysics/, or by emailing us directly at main@theexpertta.com.